MATH 54 - HINTS TO HOMEWORK 6

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Here are a couple of hints to Homework 6! Enjoy :)

SECTION 4.2: NULLSPACES, COLUMN SPACES, AND LINEAR TRANSFORMATIONS

4.2.1. Calculate $A\mathbf{w}$. If you get $\mathbf{0}$, then $\mathbf{w}$ is in the Nullspace, and if not, then it's not.

4.2.5. Just solve $A\mathbf{x} = \mathbf{0}$ and express your answer in parametric vector form.

4.2.17. (a) 4 (size of each column) (b) 2 (think number of variables of $A\mathbf{x} = \mathbf{0}$). In general, $(a)n$ and $(b)m$ where A is an $m \times n$ matrix.

4.2.21. For Nul(A), solve $A\mathbf{x} = \mathbf{0}$, for Col(A), just pick one column of A

4.2.23. (a) Can you express $\mathbf{w}$ in terms of the columns of A? (b) Is $A\mathbf{w}$ equal to $\mathbf{0}$ or not? This shows, btw, that $Nul(A)$ and $Col(A)$ can have elements in common, i.e. if $\mathbf{x}$ is in $Nul(A)$, it could also be in $Col(A)$, but for independent reasons!

4.2.25 (not on your hw, but you can try it out for fun!)

(a) **T**
(b) **F** ($\mathbf{R}^n$, think number of variables! When in doubt, write out an example of A, say $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, here A is 2×3, and $Nul(A)$ is a subspace of $\mathbb{R}^3$)
(c) **T**
(d) **T** (the book might say **F**, if it is pedantic about the fact that it didn't say 'for all $\mathbf{b}$')
(e) **T** (kernel is the same as nullspace)
(f) **T**

4.2.26.

(a) T (it's a subspace of $\mathbb{R}^n$)
(b) T (when in doubt on an exam, write out an example of A, say $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, here A is 2×3, and $Nul(A)$ is a subspace of $\mathbb{R}^2$)
(c) F (it's the set of $A\mathbf{x}$, where $\mathbf{x}$ is in $\mathbb{R}^n$
(d) T (kernel is another name for nullspace)
(e) T
(f) T (for example, if $T(y) = y'' + 2y'$, then $Ker(T)$ is the set of solutions to $y'' + 2y' = 0$)

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4.2.29.

(a) $A\mathbf{0} = \mathbf{0} = A\mathbf{x}$, where $\mathbf{x} = \mathbf{0}$
(b) $A\mathbf{u} + A\mathbf{v} = A(\mathbf{u} + \mathbf{v}) = A\mathbf{x}$, where $\mathbf{x} = \mathbf{u} + \mathbf{v}$
(c) $cA\mathbf{u} = A(c\mathbf{u}) = A\mathbf{x}$, where $\mathbf{x} = c\mathbf{u}$

4.2.32.

(a) Suppose $T(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then if $p(t) = a + bt + ct^2$, you get $p(0) = 0$, hence $a = 0$, and $p(t) = bt + ct^2$, which shows that $Ker(T) = Span\left\{t, t^2\right\}$

(b) Notice that if $p(t) = a + bt + ct^2$, then $T(p) = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $Ran(T) = Span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. (To be 100 percent rigorous, you would have to show that any vector of the form $\begin{bmatrix} a \\ a \end{bmatrix}$ is in fact in the range, but for this consider the polynomial p(t) = a)

SECTION 4.5: THE DIMENSION OF A VECTOR SPACE

4.5.1, 4.5.3. First express the subspace as the span of some vectors, and then show that the resulting vectors are linearly independent. To find the dimension, just count the number of vectors in that basis.

4.5.9. In other words, find the dimension of $V = \left\{\begin{bmatrix} x \\ y \\ x \end{bmatrix}, x, y \text{in} \mathbb{R}\right\}$

4.5.10. Notice that the second vector is a multiple of the first, so you can ignore it. On the other hand, the vectors that are left (the first and the third one) are linearly independent, hence they form a basis for H, and hence H has dimension 2. In other words, $H = \mathbb{R}^2$.

4.5.11. Group the vectors into a matrix A, row-reduce A and count the number of pivots. The dimension is equal to the number of pivots. The reason this works is because you're basically trying to figure out the dimension of $Col(A)$, and this dimension is equal to the number of pivots.

4.5.13, 4.5.15. To find $dim(Nul(A))$, first find $Nul(A)$ and for this, just solve $A\mathbf{x} = \mathbf{0}$. **CAREFUL:** You might want to continue row-reducing the matrices until you get the RREF, otherwise the algebra is a mess! For $dim(Col(A))$, it's easier, it's equal to the number of pivots!

4.5.19.

(a) T
(b) F (it has to go through $(0, 0, 0)$, otherwise it's not a subspace)
(c) F (5)
(d) (S has to have n elements, for example $S = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ is not a basis for $\mathbb{R}^3$)

(e) T (That's one of the awesome facts about dimension)

4.5.20.

(a) F (it's not even a **subset** of $\mathbb{R}^3$!!!)
(b) F (Consider $A = I_3$ (the 3×3 identity matrix), then $Nul(A) = \{\mathbf{0}\}$ because A is invertible, yet the number of variables in $A\mathbf{x} = \mathbf{0}$ is 3)
(c) F (it's an infinite linearly independent set! For example $Span\{1, 2, 3, \cdots\} = Span\{1\}\ \mathbb{R}$, yet $\mathbb{R}$ is 1-dimensional)
(d) F (S has to have n elements, for example $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is not a basis for $\mathbb{R}^3$)
(e) T (see 4.5.26)

4.5.25. Suppose $SpanS = V$. If S is linearly independent, then S is a basis of V, and hence $n = dim(V) =$ number of vectors in $S < n$, contradiction. If S is linearly dependent, we can find extract a linearly independent subset H of S by removing redundant vectors with $SpanH = SpanS = V$. Then H is a basis for V and hence $n = dim(V) =$ number of vectors in H ¡ number of vectors in $S < n$, again a contradiction.

4.5.26. Suppose $\mathcal{B} = \{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ is a basis for H. Then $\mathcal{B}$ is linearly independent and has $n = dim(V)$ vectors, hence $\mathcal{B}$ is a basis for V. Hence $H = Span\{\mathbf{v_1}, \cdots, \mathbf{v_n}\} = V$.

4.5.27. Find an infinite linearly independent set in $\mathbb{P}$. For example, $\left\{1, x, x^2 \cdots\right\}$ works!

SECTION 4.6: THE RANK OF A MATRIX

Remember that the rank of A is just $dim(Col(A))$. It is also equal to $dim(Row(A))$ and to $Rank(A^T)$ and to the number of pivots of A.

4.6.1, 4.6.3. Use the following method:

- $Rank(A)$ is the number of pivots of A
- $dim(Nul(A)) + Rank(A) = n$ (number of columns of A)
- To find a basis for $Col(A)$, row-reduce A and find the pivot columns of A (say the first, third, and fourth column for example) and then the first, third, and fourth column of A (!) is a basis for $Col(A)$.
- For a basis for $Row(A)$, row-reduce A and find the pivot rows of A (say the first and the third row). Then the first and the third row of the *row-reduced* matrix B is a basis for $Row(A)$.
- Finally, for a basis for $Nul(A)$, solve the equation $A\mathbf{x} = \mathbf{0}$. For this, use the *reduced row-echelon form*!!!

4.6.5, 4.6.7, 4.6.9, 4.6.15. Use the equation $dim(Nul(A)) + Rank(A) = n$. Also, $rank(A^T) = rank(A)$. Careful about 4.6.7, Nul(A) is a subspace of $\mathbb{R}^7$, so it *cannot* equal to $\mathbb{R}^3$. Finally, for 4.6.15, $Nul(A)$ is smallest when $Rank(A)$ is largest!

4.6.17.

(a) **T**
(b) **F** (of B!!!)
(c) **T** (the magic of linear algebra!)
(d) **F** (Columns)
(e) **F** (don't worry about this)

4.6.18.

(a) **F** (you really have to go back to the *original* matrix A to figure out a basis for $Col(A)$)
(b) **F** (they preserve the span, but not linear independence)
(c) **T** ($dim(Nul(A)) + Rank(A) = n$)
(d) **T** (try to convince yourself that this is true! Remember that to get A^T from A, you just flip the rows and columns of A)
(e) **T**

4.6.28. For (a), $dim(Row(A))$ = number of pivots = $dim(Col(A))$. For (b), use (a), but replace A with A^T. Then the number of columns of A^T equals to the number of rows of A, which is m, and also $Row(A^T) = Col(A)$

SECTION 4.7: CHANGE OF BASIS

Remember: To change coordinates from $\mathcal{B}$ to $\mathcal{C}$, just express the old vectors in $\mathcal{B}$ in terms of the vectors in the new and cool basis $\mathcal{C}$

4.7.1.

(a) The first column of P is $[\mathbf{b_1}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and the second column is $[\mathbf{b_1}]_{\mathcal{C}} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$, hence: $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$
(b) Use $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}} [\mathbf{x}]_{\mathcal{B}}$. Here $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

4.7.3. (ii), because P is just $\mathcal{W} \xleftarrow{P} \mathcal{U}$, so P goes from $\mathcal{U}$ to $\mathcal{V}$.

4.7.5. This is very similar to $4.7.1$! Look at the above hint!

4.7.11.

(a) **F** (the $\mathcal{C}$ coordinate vectors of $\mathcal{B}$, look at $4.7.1$. In other words, you take the old vectors in $\mathcal{B}$ and evaluate them with respect to the new and cool code $\mathcal{C}$)
(b) **T** (Yes, because the code of $\mathbf{x}$ in the basis $\mathcal{C}$ is just $\mathbf{x}$, i.e. $[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}$)

4.7.12.

(a) **T** (the matrix is invertible, hence by the IMT its columns must be linearly independent)
(b) **F** (no, it produces $P_{\mathcal{C}\leftarrow\mathcal{B}}$, i.e. a matrix such that $[\mathbf{x}]_{\mathcal{C}} = P\,[\mathbf{x}]_{\mathcal{B}}$))

4.7.13. First, find the codes of each polynomial, i.e. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ etc. Then group all of them in a matrix $P = P_{\mathcal{C}\leftarrow\mathcal{B}}$. Now we want to find $[p]_{\mathcal{B}}$, where $p(t) = -1 + 2t$. We know that $[p]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Now use the fact that $[\mathbf{p}]_{\mathcal{C}} = P\,[\mathbf{p}]_{\mathcal{B}})$, so $[\mathbf{p}]_{\mathcal{B}} = P^{-1}\,[\mathbf{p}]_{\mathcal{C}})$